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The Highest Common Factor of a System of Polynomials in One Variable.[*]

BY LLOYD L. DINES.

As is well known, the condition for the existence of a common factor of first or higher degree of two polynomials

$$F_1(x) \equiv c_{10}x^{m_1} + c_{11}x^{m_1-1} + \ldots . + c_{1m_1},$$
$$F_2(x) \equiv c_{20}x^{m_2} + c_{21}x^{m_2-1} + \ldots . + c_{2m_2}$$

is the vanishing of a certain rational integral function of the coefficients called the *resultant.* This resultant of two polynomials has been expressed in many different forms, and exhaustively studied.[†]

The object of the present paper is to consider the more general problem: the conditions which characterize the existence of a common factor of first or higher degree of a system

$$F_j(x) \equiv c_{j0}x^{m_j} + c_{j1}x^{m_j-1} + \ldots . + c_{jm_j} \quad (j = 1, 2, \ldots ., n),$$

consisting of any finite number of polynomials in one variable. This problem has been treated briefly by Meyer,[‡] and under certain restrictions by Foethke.[§]

Meyer showed by a generalization of the Euclidian algorithm, that there always exist $n-1$ rational integral functions of the coefficients whose vanishing constitutes necessary and sufficient conditions for the existence of a common factor of first or higher degree of the polynomials $F_1, F_2, \ldots ., F_n$. These functions appear as remainders in a process of repeated division. Foethke, by a careful analysis of the Meyer process, showed the existence of $n-1$ *determinants* such that if certain restrictions be made upon the coefficients of the polynomials $F_1, F_2, \ldots ., F_n$, the vanishing of these $n-1$ determinants is equivalent to the vanishing of the $n-1$ remainder functions of Meyer, and therefore constitutes necessary and sufficient conditions.

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[*] Read before the American Mathematical Society, October 28, 1911.

[†] By Euler, Bezout, Sylvester, Scheibner, Meyer and others. For bibliography, see *Encyclopédie des Sciences Math.*, I, 2, 73.

[‡] Franz Meyer: *Jahresbericht der Math. Ver.*, Vol. XVI (1907), p. 16.

[§] Foethke: Dissertation, Königsberg (1907), and *Archiv der Math. und Physik,* Vol. XVII (1910), p. 170.

In the present paper (Chapter II), a matrix is constructed whose elements are the coefficients c_{jk} of the given polynomials, arranged according to a simple rule, and which for $n-2$ reduces to the matrix of the ordinary resultant determinant of Euler and Sylvester for two polynomials. This matrix has the following properties:

(1) Its vanishing* constitutes necessary and sufficient conditions for the existence of a common factor of first or higher degree (Theorem I).

(2) Its rank determines the degree of the highest common factor (Theorem III).

(3) The coefficients of the highest common factor are determinants of this matrix, easily characterized (Theorem IV).

In Chapter III it is shown that in any numerical case there are $n-1$ "essential" determinants of this resultant matrix, whose simultaneous vanishing is equivalent to the vanishing of the matrix (Theorem VII). In the case $n=3$, the two $(=n-1)$ "essential" determinants are easily characterized (Theorem V). For *any* value of n, the "essential" determinants are easily characterized whenever there is a pair of the given polynomials whose coefficients satisfy a certain condition (the analytic condition that the degree of the highest common factor of the two be not higher than *one*). (Theorem VI.)

Chapter I contains two simple lemmas concerning matrices, which facilitate the succeeding proofs; and Chapter IV gives as an illustrative example a detailed treatment of the case of three cubics.

CHAPTER I.

A CERTAIN TRANSFORMATION OF MATRICES INTO EQUIVALENT MATRICES.

§ 1. *Equivalent Matrices. The Transformation T_a.*

Two matrices are said to be equivalent if one of them can be obtained from the other by a transformation which does not change the rank of a matrix. The elementary transformations which do not change the rank of a matrix are:

(1) The interchange of two rows (or columns).

(2) The multiplication of all the elements of one row (or column) by the same quantity different from zero.

(3) The addition of the elements of one row (or column) to the corresponding elements of another row (or column).

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* Throughout this paper a matrix of μ rows and ν columns $(\mu \leqq \nu)$ will be said to "vanish" when all the determinants of order μ of the matrix vanish. This definition, given by Pascal ("Die Determinanten," p. 192), though not in as general use as another which is current in the theory of matrices, is more convenient for present purposes.

In the proofs of theorems which follow, considerable use will be made of a transformation which is a repeated product of the elementary transformations (2) and (3), and which we now define.

Definition. If $\alpha_1, \alpha_2, \ldots\ldots, \alpha_h$ are any h quantities, and α denotes the sequence $(\alpha_1, \alpha_2, \ldots\ldots, \alpha_h)$, then T_α is that transformation which changes a matrix

$$K = \begin{Vmatrix} a_{11} & a_{21} & \cdots\cdots & a_{\nu 1} \\ a_{12} & a_{22} & \cdots\cdots & a_{\nu 2} \\ \cdot & \cdot & \cdots\cdots & \cdot \\ \cdot & \cdot & \cdots\cdots & \cdot \\ \cdot & \cdot & \cdots\cdots & \cdot \\ a_{1\mu} & a_{2\mu} & \cdots\cdots & a_{\nu\mu} \end{Vmatrix} \quad \text{into} \quad K' = \begin{Vmatrix} a'_{11} & a'_{21} & \cdots\cdots & a'_{\nu 1} \\ a'_{12} & a'_{22} & \cdots\cdots & a'_{\nu 2} \\ \cdot & \cdot & \cdots\cdots & \cdot \\ \cdot & \cdot & \cdots\cdots & \cdot \\ \cdot & \cdot & \cdots\cdots & \cdot \\ a'_{1\mu} & a'_{2\mu} & \cdots\cdots & a'_{\nu\mu} \end{Vmatrix}$$

where the rows of K' are found successively from the rows of K by means of the recursion formulas

$$a'_{jk} = a_{jk} - \sum_{i=1}^{h} \alpha_i \, a'_{j\,k-i} \qquad (j = 1, 2, \ldots\ldots, \nu),$$

$a'_{j\,k-i}$ $(i \geqq k)$ being interpreted as zero. The relation existing between K and K' is expressed symbolically by

$$T_\alpha K = K'.$$

Evidently K' can be obtained from K by repetition of the transformations (2) and (3), and we therefore have

LEMMA I. *If two matrices M and M' are so related that there exists a sequence of quantities $\alpha = (\alpha_1, \alpha_2, \ldots\ldots, \alpha_h)$ such that $T_\alpha M = M'$, then the two matrices are equivalent.*

§ 2. *Application of the Transformation T_α to a Particular Type of Matrices.*

In what follows, the transformation T_α is applied to a particular type of matrices, namely those of the form[*]

$$M = \begin{array}{c} \multicolumn{1}{c}{N - m \text{ columns}} \\ \begin{Vmatrix} c_0 & & & & \\ c_1 & c_0 & & & \\ \cdot & c_1 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & c_0 \\ \cdot & \cdot & \cdot & \cdot & c_1 \\ c_m & \cdot & \cdot & \cdot & \cdot \\ & c_m & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & c_m \end{Vmatrix} \end{array} \; N \text{ rows} \qquad (1)$$

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[*] In the representation of this matrix, and likewise of others which follow, the portions of the array in which no elements are indicated, either by letters or by dots, contain only zero elements.

in which each column contains as elements the sequence of quantities c_0, c_1, ..., c_m, and zeros.

Frequent use is made of the following

LEMMA II. *If for any sequence* $\alpha = (\alpha_1, \alpha_2, \ldots., \alpha_h)$, *the matrix* M *of* (1) *is changed by the transformation* T_α *into*

$$M' = \begin{array}{cc} \text{$N-m$ columns} & \\ \left\| \begin{array}{ccccc} c'_0 & & & & \\ c'_1 & c'_0 & & & \\ \cdot & c'_1 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & c'_0 \\ \cdot & \cdot & \cdot & \cdot & c'_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c'_{N-h} & \cdot & \cdot & \cdot & c'_{m-h+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c'_{N-1} & c'_{N-2} & \cdot & \cdot & c'_m \end{array} \right\| & \text{N rows} \end{array}$$

then the matrix consisting of the last h *rows of* M' *vanishes, if the determinant*

$$\left| \begin{array}{cccc} c'_m & c'_{m-1} & \cdot & c'_{m-h+1} \\ c'_{m+1} & c'_m & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ c'_{m+h-1} & \cdot & \cdot & c'_m \end{array} \right| \qquad (2)$$

consisting of the elements common to the last h *rows and last* h *columns of* M' *vanishes.*

To prove this lemma, we note that by the definition of the transformation T_α we have the relations

$$c'_k = c_k - \sum_{i=1}^{h} \alpha_i c'_{k-i} \qquad (k = 0, 1, \ldots., N-1), \qquad (3)$$

where the notations c_k $(k > m)$ and c'_{k-i} $(i > k)$ are interpreted as zero.

Now since the elements of any one of the last h rows of M' constitute some segment of the sequence

$$c'_{N-1},\ c'_{N-2},\ c'_{N-3},\ \ldots.,\ c'_{m-h+1},$$

it is evident from (3) that if to one of the first $N-m-h$ elements, say the

p-th element, of such a row, there be added the $(p+1)$-th, $(p+2)$-th, ..., $(p+h)$-th elements multiplied respectively by $\alpha_1, \alpha_2, \ldots, \alpha_h$, the result is the corresponding element of the matrix M. Therefore, by processes which do not change the rank of a matrix, the matrix consisting of the last h rows of M' can be transformed into one in which the elements of the last h columns remain unchanged, while the elements of the other columns are the corresponding elements of the matrix consisting of the last h rows of M. But these elements of M are all zero, and from this it follows that the matrix consisting of the last h rows of M' vanishes if the determinant (2) vanishes.

CHAPTER II.

THE RESULTANT MATRIX.

§ 3. *Conditions for the Existence of a Common Factor.*

Given the system of polynomials

$$F_j(x) = c_{j0}x^{m_j} + c_{j1}x^{m_j-1} + \ldots + c_{jm_j} \quad (j = 1, 2, \ldots, n), \tag{4}$$

where the coefficients c_{jk} are independent of x, and the leading coefficients c_{j0} are all different from zero. It will further be supposed that the polynomials F_j are arranged according to their degrees, so that $m_i \geqq m_{i-1}$.

Of fundamental importance in the theory of the highest common factor of the polynomials (4) is a matrix constructed as follows:

$$M_N(F_1 \ldots F_n) = \begin{array}{cccccccccccccccc}
 & \multicolumn{4}{c}{\overbrace{\qquad N - m_1 \text{ columns} \qquad}} & & \multicolumn{4}{c}{\overbrace{\qquad N - m_j \text{ columns} \qquad}} & & \multicolumn{4}{c}{\overbrace{\qquad N - m_n \text{ columns} \qquad}} & & \\
\Bigg\| & c_{10} & & & & \cdots & c_{j0} & & & & \cdots & c_{n0} & & & & \Bigg\| \\
\Bigg\| & c_{11} & c_{10} & & & \cdots & c_{j1} & c_{j0} & & & \cdots & c_{n1} & c_{n0} & & & \Bigg\| \\
\Bigg\| & \cdot & c_{11} & \cdot & & \cdots & \cdot & c_{j1} & \cdot & & \cdots & \cdot & c_{n1} & \cdot & & \Bigg\| \\
\Bigg\| & \cdot & \cdot & \cdot\;\cdot & & \cdots & \cdot & \cdot & \cdot\;\cdot & & \cdots & \cdot & \cdot & \cdot\;\cdot & & \Bigg\| \\
\Bigg\| & \cdot & \cdot & \cdot\;\cdot & c_{10} & \cdots & \cdot & \cdot & \cdot\;\cdot & c_{j0} & \cdots & \cdot & \cdot & \cdot\;\cdot & c_{n0} & \Bigg\| \; N \text{ rows} \\
\Bigg\| & c_{1m_1} & \cdot & \cdot\;\cdot & c_{11} & \cdots & c_{jm_j} & \cdot & \cdot\;\cdot & c_{j1} & \cdots & c_{nm_n} & \cdot & \cdot\;\cdot & c_{n1} & \Bigg\| \\
\Bigg\| & & c_{1m_1} & \cdot\;\vdots & \cdot & \cdots & & c_{jm_j} & \cdot\;\cdot & \cdot & \cdots & & c_{nm_n} & \cdot\;\cdot & \cdot & \Bigg\| \\
\Bigg\| & & & \cdot\;\cdot & \cdot & \cdots & & & \cdot\;\cdot & \cdot & \cdots & & & \cdot\;\cdot & \cdot & \Bigg\| \\
\Bigg\| & & & \cdot & \cdot & \cdots & & & \cdot & \cdot & \cdots & & & \cdot & \cdot & \Bigg\| \\
\Bigg\| & & & & c_{1m_1} & \cdots & & & & c_{jm_j} & \cdots & & & & c_{nm_n} & \Bigg\|
\end{array}$$

where N is any integer satisfying the conditions

$$N \geqq m_i + m_j \quad (i \neq j). \tag{5}$$

The matrix $M_N(F_1 \ldots F_n)$ has N rows and $\sum\limits_{j=1}^{n}(N - m_j)$ columns, the latter being arranged in n groups, each group containing elements which are the coefficients of one polynomial F_j. The group corresponding to the polynomial F_j has $N - m_j$ columns and will be denoted by G_j.

There are evidently an infinite number of such matrices corresponding to the infinite number of possible values of N, the matrix of lowest possible order being given by $N = m_1 + m_2$. Any matrix $M_N(F_1 \ldots . F_n)$ constructed in the manner indicated is called a *resultant matrix* of the system (4), The theory which follows is developed for the *general* resultant matrix $M_N(F_1 \ldots . F_n)$, N being any integer satisfying the relations (5).

THEOREM I. *The vanishing* of the resultant matrix* $M_N(F_1 \ldots . F_n)$ *is necessary and sufficient for the existence of a common factor of first or higher degree of the polynomials* F_j.

The necessity of the condition may be seen as follows: If the polynomials F_j have a common factor containing x, then the n equations $F_j = 0$ $(j = 1, 2, \ldots ., n)$ have at least one common root, and therefore the system of equations

$$\left.\begin{array}{lr} x^{N-m_j-1} \cdot F_j \equiv c_{j0}\, x^{N-1} + c_{j1}\, x^{N-2} + \ldots + c_{jm_j}\, x^{N-m_j-1} & = 0, \\ x^{N-m_j-2} \cdot F_j \equiv \qquad c_{j0}\, x^{N-2} + c_{j1}\, x^{N-3} + \ldots + c_{jm_j}\, x^{N-m_j-2} & = 0, \\ \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot\quad \cdot & \cdot\,, \\ x \cdot F_j \equiv \qquad\qquad c_{j0}\, x^{m_j+1} + c_{j1}\, x^{m_j} + \ldots + c_{j0}\, x & = 0, \\ F_j \equiv \qquad\qquad\qquad c_{j0}\, x^{m_j} + c_{j1}\, x^{m_j-1} + \ldots + c_{jm_j} & = 0 \\ \qquad\qquad (j = 1, 2, \ldots ., n), & \end{array}\right\} \quad (6)$$

considered as a system of linear homogeneous equations in the quantities $x^{N-1}, x^{N-2}, \ldots ., x, 1$, has a solution. From the theory of linear equations it follows that the matrix of coefficients of this system of equations must vanish, and that is exactly the matrix $M_N(F_1 \ldots . F_n)$.

To prove the sufficiency of the condition, we employ a device used by Kronecker in his general theory of elimination. If u_k $(k = 1, 3, 4, \ldots ., n)$ and u'_l $(l = 2, 3, \ldots ., n)$ are two sets of arbitrary parameters, then the two polynomials

$$\sum_k F_k\, u_k, \quad \sum_l F_l\, u'_l$$

have a common factor containing x, only if the polynomials F_j have such a common factor. But, as is well known, these two polynomials have a common factor containing x if their resultant determinant vanishes, and therefore the vanishing of this determinant forms a sufficient condition for the existence of a common factor of the polynomials F_j. Since the polynomials $\sum_k F_k\, u_k$ and $\sum_l F_l\, u'_l$ are respectively of degrees m_1 and m_2 in x, their resultant determinant is of order $m_1 + m_2$. It is furthermore a rational integral function of the parameters u_k and u'_l, and can therefore be expanded in the form

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* See foot-note, p. 130, concerning definition of term "vanish."

$$\rho_0 U_0 + \rho_1 U_1 + \ldots . ,$$

where $U_0, U_1, \ldots .$ are different power products of the parameters u_k and u'_l; and $\rho_0, \rho_1, \ldots .$ are functions of the coefficients of the polynomials F_j. The polynomials F_j have therefore a common factor containing x if the functions $\rho_0, \rho_1, \ldots .$ all vanish.

Now if the resultant determinant of the two polynomials $\sum\limits_k F_k u_k$ and $\sum\limits_l F_l u'_l$ be constructed, it is seen that the functions $\rho_0, \rho_1, \ldots .$ are linear combinations of determinants of order $m_1 + m_2$ of a matrix

$$\begin{array}{cccccccccccccccccccc}
 & & & m_2\ \text{columns} & & & & & & m_1\ \text{columns} & & & & & & m_1\ \text{columns} & & & & \\
\Vert & c_{10} & & & & & \cdots & & & & & & \cdots & & & & & & \Vert & \\
\Vert & c_{11} & c_{10} & & & & \cdots & c_{j0} & & & & & \cdots & & & & & & \Vert & \\
\Vert & \cdot & c_{11} & \cdot & & & \cdots & c_{j1} & c_{j0} & & & & \cdots & c_{n0} & & & & & \Vert & \\
\Vert & \cdot & \cdot & \cdot & \cdot & & \cdots & \cdot & c_{j1} & \cdot & & & \cdots & c_{n1} & c_{n0} & & & & \Vert & \\
\Vert & \cdot & \cdot & \cdot & \cdot & c_{10} & \cdots & \cdot & \cdot & \cdot & \cdot & & \cdots & \cdot & c_{n1} & \cdot & & & \Vert & \\
\Vert & \cdot & \cdot & \cdot & \cdot & c_{11} & \cdots & \cdot & \cdot & \cdot & \cdot & c_{j0} & \cdots & \cdot & \cdot & \cdot & \cdot & & \Vert & m_1+m_2\ \text{rows} \\
\Vert & c_{1m_1} & \cdot & \cdot & \cdot & \cdot & \cdots & c_{jm_j} & \cdot & \cdot & \cdot & c_{j1} & \cdots & c_{nm_n} & \cdot & \cdot & \cdot & c_{n0} & \Vert & \\
\Vert & & c_{1m_1} & \cdot & \cdot & \cdot & \cdots & & c_{jm_j} & \cdot & \cdot & \cdot & \cdots & & c_{nm_n} & \cdot & \cdot & c_{n1} & \Vert & \\
\Vert & & & \cdot & \cdot & \cdot & \cdots & & & \cdot & \cdot & \cdot & \cdots & & & \cdot & \cdot & \cdot & \Vert & \\
\Vert & & & & \cdot & \cdot & \cdots & & & & \cdot & \cdot & \cdots & & & & \cdot & \cdot & \Vert & \\
\Vert & & & & & c_{1m_1} & \cdots & & & & & c_{jm_j} & \cdots & & & & & c_{nm_n} & \Vert &
\end{array} \tag{7}$$

Therefore if the matrix (7) vanishes, the functions $\rho_0, \rho_1, \ldots .$ are all equal to zero.

But the matrix (7) vanishes whenever the matrix $M_N(F_1 \ldots . F_n)$ vanishes. For, corresponding to every determinant Δ of order $m_1 + m_2$ in (7), there is a determinant of order N in $M_N(F_1 \ldots . F_n)$ which is equal to $c_{10}^{N-(m_1+m_2)} \cdot \Delta$, and since c_{10} is different from zero, this determinant can not vanish unless Δ does. Therefore if the matrix $M_N(F_1 \ldots . F_n)$ vanishes, the functions $\rho_0, \rho_1, \ldots .$ are all equal to zero, and the polynomials F_j have a common factor containing x.

§ 4. *Sub-resultant Matrices. Degree of the Highest Common Factor.*

It has been shown in the preceding section that the vanishing of the resultant matrix $M_N(F_1 \ldots . F_n)$ is necessary and sufficient for the existence of a common factor of first or higher degree of the polynomials (4). In the present section it will be shown that the exact degree of the highest common factor can be stated in terms of the rank of the resultant matrix, the final result being reached in Theorem III. To this end we define a certain sequence of matrices called *sub-resultant matrices* and denoted by

$$M_N^{(i)}(F_1 \ldots . F_n) \quad (i = 1, 2, \ldots ., m_n).$$

DEFINITION. *$M_N^{(i)}(F_1 \ldots . F_n)$ is the matrix formed from $M_N(F_1 \ldots . F_n)$ by striking out the first i rows and the last i rows, and the first i columns of each group G_j.* It has therefore $N - 2i$ rows and $\sum_{j=1}^{n} (N - m_j - i)$ columns.

THEOREM II. *The highest common factor of the polynomials* (4) *is of degree k ($\leqq m_n$), if and only if*

$$M_N(F_1 \ldots . F_n) = 0; \quad M_N^{(i)}(F_1 \ldots . F_n) = 0 \quad (i = 1, 2, \ldots ., k-1), \tag{8}$$

while

$$M_N^{(k)}(F_1 \ldots . F_n) \neq 0. \tag{9}$$

The proof of this theorem is accomplished by means of the following:

LEMMA III. *If the polynomials F_j have a common factor*

$$D^{(h)}(x) \equiv x^h + \alpha_1 x^{h-1} + \alpha_2 x^{h-2} + \ldots . + \alpha_h,$$

and if

$$F_j \equiv D^{(h)} \cdot F_j^{(h)} \quad (j = 1, 2, \ldots ., n), \tag{10}$$

then:

(1) *The matrix $T_\alpha M_N(F_1 \ldots . F_n)$, where α is the sequence $(\alpha_1, \alpha_2, \ldots ., \alpha_h)$ of coefficients of $D^{(h)}$, has only zero elements in the last h rows.*

(2) *The sub-resultant matrix $M_N^{(h)}(F_1 \ldots . F_n)$ is equivalent to the resultant matrix $M_{N-2h}(F_1^{(h)} \ldots . F_n^{(h)})$ of the quotient polynomials $F_j^{(h)}$.*

For the proof of the lemma, let any one of the polynomials F_j be denoted temporarily by

$$F \equiv c_0 x^m + c_1 x^{m-1} + \ldots . + c_m,$$

and the corresponding quotient polynomial $F/D^{(h)}$ by

$$F^{(h)} \equiv c_0' x^{m-h} + c_1' x^{m-h-1} + \ldots . + c_{m-h}'.$$

Then the relations existing between the coefficients of F, $D^{(h)}$ and $F^{(h)}$, on account of (10), are

$$c_k' = c_k - \sum_{i=1}^{h} \alpha_i c_{k-i}' \quad (k = 0, 1, \ldots ., m), \tag{11}$$

where the notations c_k' $(k > m - h)$ and c_{k-i}' $(i > k)$ are to be interpreted as zero.

From these relations and the definition of the transformation T_α it follows that the matrix

$$\begin{array}{cc} N-m\ \text{columns} & \\ \left\|\begin{array}{ccccc} c_0 & & & & \\ c_1 & c_0 & & & \\ \cdot & c_1 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot & c_0 \\ \cdot & \cdot & \cdot & \cdot & c_1 \\ c_m & \cdot & \cdot & \cdot & \cdot \\ & c_m & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & c_m \end{array}\right\| & N\ \text{rows} \end{array} \tag{12}$$

formed from the columns of $M_N(F_1 \ldots . F_n)$ which correspond to the polynomial F, is transformed by the transformation T_α into

$$\begin{array}{cc} N-m\ \text{columns} & \\ \left\|\begin{array}{ccccc} c'_0 & & & & \\ c'_1 & c'_0 & & & \\ \cdot & c'_1 & \cdot & & \\ \cdot & \cdot & \cdot & \cdot & \\ c'_{m-h} & \cdot & \cdot & \cdot & c'_0 \\ & c'_{m-h} & \cdot & \cdot & c'_1 \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & & c'_{m-h} \\ & & & & \\ & & & & \\ & & & & \end{array}\right\| & \begin{array}{l} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\\1\\1\\1\\1\\1\end{array}}\right\} N-h\ \text{rows} \\ \left.\vphantom{\begin{array}{c}1\\1\\1\end{array}}\right\} h\ \text{rows} \end{array} \end{array} \tag{13}$$

in which the last h rows contain only zero elements. Since, then, every matrix formed from a group G_j of columns of $M_N(F_1 \ldots . F_n)$ is transformed by T_α into a matrix which contains only zero elements in the last h rows, the same is true of the matrix $M_N(F_1 \ldots . F_n)$ itself, and this proves the first part of the lemma.

Now, the group of columns in $M_N^{(h)}(F_1 \ldots . F_n)$ which corresponds to the polynomial F is, by definition, obtained from (12) by crossing out the first and last h rows and the first h columns. The matrix formed from this group of columns is transformed by T_α into the matrix obtained by striking out the first and last h rows and the first h columns of (13). But it is evident that this latter matrix consists of exactly those columns of $M_{N-2h}(F_1^{(h)}\ldots . F_n^{(h)})$ which correspond to the polynomial $F^{(h)}$. From this follows at once

$$T_a M_N^{(h)}(F_1 \ldots . F_n) = M_{N-2h}(F_1^{(h)} \ldots . F_n^{(h)}),$$

and the second conclusion of the lemma.

We now prove Theorem II.

(a) The conditions of the theorem are necessary. For, the existence of a common factor $D^{(k)}$ of degree k of the polynomials F_j necessitates the existence of common factors of degrees 1, 2, ..., $k-1$. If these be denoted respectively by $D^{(i)}$ $(i = 1, 2, \ldots ., k-1)$, and if

$$F_j \equiv D^{(i)} F_j^{(i)} \qquad (j = 1, 2, \ldots ., n),$$

then each of the sets of polynomials $F_j^{(i)}$ (for a fixed value of i) possesses a common factor of first or higher degree. Hence, by Theorem I, each of the resultant matrices $M_{N-2i}(F_1^{(i)} \ldots . F_n^{(i)})$ vanishes; and by Lemma III the same is true of the matrices $M_N^{(i)}(F_1 \ldots . F_n)$ $(i = 1, 2, \ldots ., k-1)$.

Moreover, $M_N^{(k)}(F_1 \ldots . F_n)$ does not vanish. For if it did, the same would be true of the equivalent matrix $M_{N-2k}(F_1^{(k)} \ldots . F_n^{(k)})$, and therefore the polynomials $F_j^{(k)}$ would have a common factor. This would require that the highest common factor of the polynomials F_j be of degree greater than k, which is contrary to our assumption.

(b) The conditions of the theorem are sufficient. For, these conditions being satisfied, the polynomials F_j have, by Theorem I, a common factor of at least the first degree. They have, moreover, a common factor of degree k. For suppose the highest common factor were of degree i $(< k)$. Let it be denoted by $D^{(i)}$, and let

$$F_j = D^{(i)} F_j^{(i)} \qquad (j = 1, 2, \ldots ., n).$$

Then, by Lemma III, the resultant matrix $M_{N-2i}(F_1^{(i)} \ldots . F_n^{(i)})$ is equivalent to $M_N^{(i)}(F_1 \ldots . F_n)$, which vanishes by hypothesis, and therefore the polynomials $F_j^{(i)}$ have a common factor of at least the first degree, which contradicts the supposition that $D^{(i)}$ was the highest common factor of the polynomials F_j. The polynomials F_j have therefore a common factor of degree k.

This factor of degree k is the highest common factor. For if

$$F_j \equiv D^{(k)} F_j^{(k)} \qquad (j = 1, 2, \ldots ., n),$$

then $M_{N-2k}(F_1^{(k)} \ldots . F_n^{(k)})$ is equivalent to $M_N^{(k)}(F_1 \ldots . F_n)$, which does not vanish. Therefore the polynomials $F_j^{(k)}$ have no common factor containing x, and the highest common factor of the polynomials F_j is of degree k.

THEOREM III. *The highest common factor of the polynomials* $F_1, F_2, \ldots,$ F_n *is of degree* k, *where* $N-k$ *is the rank of the resultant matrix* $M_N(F_1 \ldots F_n)$. *The rank can not be smaller than* $N-m_n$, *where* m_n *is the smallest of* $m_1, m_2, \ldots, m_n$.

The last part of the theorem follows at once from the fact that the determinant of order $N-m_n$ formed from the elements common to the first $N-m_n$ rows and the last $N-m_n$ columns of the matrix is equal to $c_{n0}^{N-m_m}$, and is therefore different from zero.

For the proof of the first part of the theorem, it is sufficient, on account of Theorem II, to show that conditions (8) and (9) are equivalent to the conditions that the matrix $M_N(F_1 \ldots F_n)$ be of rank $N-k$.

First, if conditions (8) and (9) be satisfied, then the matrix $M_N(F_1 \ldots F_n)$ is of rank $N-k$.

It can not be of rank greater than $N-k$. For under the assumption of conditions (8), the polynomials F_j have a common factor of degree k, and the matrix $M_N(F_1 \ldots F_n)$ can, by Lemma III, be transformed into a matrix $M'_N(F_1 \ldots F_n)$ of the same rank, and such that all elements of the last k rows are zero. Since, then, $M'_N(F_1 \ldots F_n)$ is obviously of rank not greater than $N-k$, the same is true of $M_N(F_1 \ldots F_n)$.

To prove that the rank of $M_N(F_1 \ldots F_n)$ is exactly $N-k$, it suffices to note that, by condition (9), there is at least one determinant, call it K, of order $N-2k$ in $M_N^{(k)}(F_1 \ldots F_n)$, which is different from zero. Therefore the determinant of order $N-k$ formed by annexing to K those elements of the first k rows of $M_N(F_1 \ldots F_n)$ which lie in the same columns with K (all of which elements are zero), and those elements of the first k columns of $M_N(F_1 \ldots F_n)$ which lie in the first $N-k$ rows, does not vanish, for this determinant is equal to $c_{10}^k K$.

Conversely, if the matrix $M_N(F_1 \ldots F_n)$ is of rank $N-k$, then the conditions (8) and (9) are satisfied.

To prove by the indirect method that (8) is satisfied, suppose for some positive integer h, less than k, there were a determinant H of order $N-2h$ in $M_N^{(h)}(F_1 \ldots F_n)$ which did not vanish. Then the determinant of order $N-h$ formed by annexing to H those elements from the first h rows of $M_N(F_1 \ldots F_n)$ which lie in the same columns with H (which elements are all zero), and those elements of the first h columns of $M_N(F_1 \ldots F_n)$ which lie in the first $N-h$ rows, would equal $c_{10}^h H$ and therefore would not vanish. This would contradict the hypothesis that $M_N(F_1 \ldots F_n)$ is of rank k, and the contradiction proves that (8) must be satisfied.

To prove that (9) is satisfied, suppose that $M_N^{(k)}(F_1 \ldots . F_n)$ vanished. Then the polynomials F_j would have a common factor of degree $k+1$, and the matrix $M_N(F_1 \ldots . F_n)$ could, by Lemma III, be transformed into an equivalent matrix in which all elements of the last $k+1$ rows were zero. Since this latter matrix could obviously not be of rank $N-k$, the matrix $M_N(F_1 \ldots , F_n)$ could not be of that rank, as it is by hypothesis. This completes the proof of the theorem.

§ 5. *Determination of the Highest Common Factor.*

Suppose now the rank of the matrix $M_N(F_1 \ldots . F_n)$ is $N-k$, that is, the polynomials F_j have a common factor of degree k; then, as has been seen, there is a determinant of order $N-2k$ in the sub-resultant matrix $M_N^{(k)}(F_1 \ldots . F_n)$ which does not vanish.* Let this determinant be denoted by Δ_0.

In the $N-2k$ columns of $M_N(F_1 \ldots . F_n)$ which contain the columns of Δ_0, the elements of the first k rows are all zero. The matrix formed from these $N-2k$ columns by striking out the first k rows of zero elements will be denoted by M, and is a matrix of $N-k$ rows and $N-2k$ columns. According to our definition, the determinant formed from the first $N-2k$ rows of M is Δ_0. In general, the determinant whose first $N-2k-1$ rows are the first $N-2k-1$ rows of M, and whose last row is the $(N-2k+i)$-th row of M, will be denoted by Δ_i. There are $k+1$ such determinants, namely, Δ_0, Δ_1, $\ldots .$, Δ_k, and in terms of them we can state

THEOREM IV. *The highest common factor of the polynomials F_j is*

$$\Delta_0 x^k + \Delta_1 x^{k-1} + \ldots . + \Delta_k . \tag{D}$$

Since the matrix $M_N(F_1 \ldots . F_n)$ is the matrix of coefficients of the system of equations (6), it is evident that M is the matrix of coefficients of a system of equations

$$x^{i_p} \cdot F_{j_p} = 0 \quad (p = 1, 2, \ldots ., N-2k). \tag{14}$$

Now, let the co-factors of the elements of the last row of Δ_0 be denoted by C_1, C_2, $\ldots .$, C_{N-2k}; C_p being the co-factor of that element which is a coefficient in the p-th equation of (14). If the left sides of equations (14) be multiplied respectively by the corresponding co-factors C_p, and the results be added, we obtain the identity

$$\sum_{p=1}^{N-2k} C_p x^{i_p} \cdot F_{j_p} \equiv \Delta_0 x^k + \Delta_1 x^{k-1} + \ldots . + \Delta_k \equiv D, \tag{15}$$

the coefficients of the higher powers of x on the left-hand side being zero in consequence of the well-known theorem that the sum of the products of the elements of one row of a determinant by the co-factors of the corresponding elements of another row is zero.

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* Of course it is entirely possible that there be several determinants of this character. The important point is that there is one such.

From the identity (15), it follows that every common factor of the polynomials F_{j_p} $(p = 1, 2, \ldots, N - 2k)$ is a factor of the polynomial D, and therefore the highest common factor of the polynomials F_j $(j = 1, 2, \ldots, n)$, which is known to be of degree k, must be a factor of D. Therefore D must be the highest common factor.

CHAPTER III.

RESULTANT DETERMINANTS.

§ 6. *Definitions and Notations.*

In Chapter II it was shown that a necessary and sufficient condition for the existence of a common factor of first or higher degree of the polynomials (4) is the vanishing of the matrix $M_N(F_1 \ldots F_n)$. This is not in general a single condition, but many, since it means the vanishing of all the determinants of order N in the matrix $M_N(F_1 \ldots F_n)$. These determinants are not, however, independent, and from the known result of Meyer that there are $n-1$ rational integral functions of the coefficients whose vanishing constitutes necessary and sufficient conditions for the existence of a common factor, one is led to suspect that there are $n-1$ particular determinants of the matrix whose vanishing is equivalent to the vanishing of the matrix. This seems not to be strictly true, in that it seems impossible to characterize formally and generally* $n-1$ determinants of the matrix whose vanishing is equivalent to the vanishing of the matrix.

However, it will be shown in the present chapter that in every *numerical* case there are $n-1$ determinants of the matrix whose vanishing is equivalent to the vanishing of the matrix. As a rule these determinants have to be constructed successively, the form of each after the first depending upon the ranks of those which precede. However, in certain important cases, the $n-1$ determinants are easily characterized.

In order not to disturb the continuity of the discussion later, we introduce here some definitions and notations.

$M_N(F_r F_s)$. First we note that the columns of the groups G_r and G_s

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* Cf. the restrictions upon the coefficients required in Foethke's conditions, *loc. cit.*

(where r and s are any two different numbers of the set $1, 2, \ldots, n$) of the matrix $M_N(F_1 \ldots F_n)$ constitute the matrix $M_N(F_r F_s)$.

$R(F_r F_s)$. The ordinary resultant determinant (as formed by Euler and Sylvester) of the two polynomials F_r and F_s will be denoted by $R(F_r F_s)$. It is of order $m_r + m_s$ and is obtained from $M_N(F_r F_s)$ by striking out the first $N - (m_r + m_s)$ rows, the first $N - (m_r + m_s)$ columns of G_r, and the first $N - (m_r + m_s)$ columns of G_s.

$R^{(i)}(F_r F_s)$. The sub-resultant determinant $R^{(i)}(F_r F_s)$ is obtained from $R(F_r F_s)$ by striking out the first i rows, the last i rows, the first i columns containing coefficients of F_r, and the first i columns containing coefficients of F_s. It is of order $m_r + m_s - 2i$.

$P_N^{(i)}(F_r F_s)$. A determinant of order $N - i$ which is equal to

$$c_{r0}^{N-(m_r+m_s)+i} \cdot R^{(i)}(F_r F_s)$$

can evidently be formed by bordering the determinant $R^{(i)}(F_r F_s)$ on the top by the zero elements which lie above it in the matrix $M_N(F_1 \ldots F_n)$ and at the left by the first $N - (m_r + m_s) + i$ columns of the group G_r. This determinant is denoted by $P_N^{(i)}(F_r F_s)$. It vanishes if and only if $R^{(i)}(F_r F_s)$ vanishes.

$P_N^{(i)}(F_r F_s / F_j)$. The determinant of order N formed by annexing to $P_N^{(i)}(F_r F_s)$ those elements of the last i rows of $M_N(F_r F_s)$ which lie in the same columns with $P_N^{(i)}(F_r F_s)$, and to the matrix so formed annexing the last i columns of the group G_j (where j is any number of the set $1, 2, \ldots, n$, different from r and s), is denoted by $P_N^{(i)}(F_r F_s / F_j)$.

$R^{(i)}(F_r F_s / F_j)$. The determinant $P_N^{(i)}(F_r F_s / F_j)$ just defined is, from its form, equal to a product of a certain power of c_{r0} into another determinant of lower order. This latter determinant is denoted by $R^{(i)}(F_r F_s / F_j)$. Since it plays an important rôle, we give the following more direct method of constructing* it.

To the determinant $R^{(i)}(F_r F_s)$ annex those elements of the last i rows of $M_N(F_r F_s)$ which lie in the same columns with $R^{(i)}(F_r F_s)$. The resulting matrix has $m_r + m_s - i$ rows and $m_r + m_s - 2i$ columns. To the right of this matrix annex i columns consisting of those elements in the last i columns and the last $m_j + i$ rows of the group G_j. If $m_j + i = m_r + m_s - i$, the result is a square matrix of order $m_r + m_s - i$, and the corresponding determinant is $R^{(i)}(F_r F_s / F_j)$. If $m_j + i < m_r + m_s - i$, fill in the upper right-hand corner with zeros until a square matrix of order $m_r + m_s - i$ is obtained. The corresponding determinant is $R^{(i)}(F_r F_s/F_j)$. If $m_j + i > m_r + m_s - i$, prefix those elements of G_r which lie in the $m_j + i - (m_r + m_s - i)$ columns immediately preceding $R^{(i)}(F_r F_s)$ and in the last $m_j + i$ rows. By filling in above $R^{(i)}(F_r F_s)$ with zeros, a square matrix of order $m_j + i$ is then obtained, whose corresponding determinant is $R^{(i)}(F_r F_s/F_j)$.

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* The determinants $R^{(1)}(F_1 F_2 / F_3)$ and $R^{(2)}(F_1 F_2 / F_3)$, where F_1, F_2 and F_3 are cubics, are displayed in Chapter IV, and will serve as illustrations of the constructions here described.

§ 7. *The Case* $n = 2$.

If the number of polynomials considered is two, and if N be taken equal to $m_1 + m_2$, then $M_N(F_1 F_2)$ is the matrix of the resultant determinant $R(F_1 F_2)$, and the sub-resultant matrices $M_N^{(i)}(F_1 F_2)$ are respectively the matrices of the determinants $R^{(i)}(F_1 F_2)$. We have, therefore, as special cases of Theorems II and III, the two well-known theorems:

The highest common factor of two polynomials F_1 *and* F_2 *is of degree* k, *if and only if* $R(F_1 F_2)$ *and* $R^{(i)}(F_1 F_2)$ $(i = 1, 2, \ldots, k-1)$ *all vanish, while* $R^{(k)}(F_1 F_2)$ *does not vanish.*[*]

The highest common factor of two polynomials F_1 *and* F_2 *is of degree* k, *where* $N - k$ *is the rank of the resultant determinant* $R(F_1 F_2)$.[†]

§ 8. *The Case* $n = 3$.

Regarding three polynomials

$$\left.\begin{aligned} F_1(x) &\equiv a_0 x^{\lambda} + a_1 x^{\lambda-1} + \ldots + a_{\lambda}, \\ F_2(x) &\equiv b_0 x^{\mu} + b_1 x^{\mu-1} + \ldots + b_{\mu}, \\ F_3(x) &\equiv c_0 x^{\nu} + c_1 x^{\nu-1} + \ldots + c_{\nu}, \end{aligned}\right\} \qquad (16)$$

we have the following

THEOREM V. *If* F_r *and* F_s *are any two of the three polynomials* (16), *and if* $R^{(i)}(F_r F_s) = 0$ $(i = 1, 2, \ldots, k-1)$, *but* $R^{(k)}(F_r F_s) \neq 0$, *then necessary and sufficient conditions for the vanishing of the matrix* $M_N(F_1 F_2 F_3)$, *and therefore for the existence of a common factor of first or higher degree, are*

$$R(F_r F_s) = 0, \quad R^{(k)}(F_r F_s/F_j) = 0 \quad (j \neq r, s). \qquad (17)$$

That the conditions are necessary is evident since the determinants $R(F_r F_s)$ and $R^{(k)}(F_r F_s/F_j)$ are, when multiplied by certain powers of a_0, equal to determinants of order N in $M_N(F_1 F_2 F_3)$.

Assuming now that $R(F_r F_s) = 0$, it follows from the hypothesis of the theorem that the highest common factor of the polynomials F_r and F_s is of degree k. Therefore by Lemma III of Chapter II, the matrix $M_N(F_1 F_2 F_3)$ can be transformed by a transformation T_a into an equivalent matrix $M'_N(F_1 F_2 F_3)$

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[*] Cf. Scheibner: "Mathematische Bemerkungen," *Leipz. Ber.*, 40, pp. 1–13.

[†] Cf. Capelli: *Palermo Rend.*, 23, pp. 130–136.

in which the last k rows of $M'_N(F_r F_s)$ contain only zero elements. By this same transformation the determinant $P_N^{(k)}(F_r F_s)$, which is equal to

$$a_0^{N-(m_r+m_s-k)} \cdot R^{(k)}(F_r F_s)$$

and is therefore different from zero, is changed into a determinant $P_N^{(k)\prime}(F_r F_s)$ which does not vanish. If, for the sake of definiteness in writing, we suppose $r = 1$, $s = 2$, then $M'_N(F_1 F_2 F_3)$ will be of the form

$$M'_N(F_1 F_2 F_3) = \begin{array}{||cccccccccccc||l}
\overbrace{\hphantom{a'_{\lambda-k}\;a'_{\lambda-k}\;\cdot\cdot\;a'_0}}^{N-\lambda\text{ columns}} & & & & \overbrace{\hphantom{b'_{\mu-k}\;b'_{\mu-k}\;\cdot\cdot\;b'_0}}^{N-\mu\text{ columns}} & & & & \overbrace{\hphantom{c'_{N-k}\;c'_{N-k-1}\;\cdot\cdot\;c'_{\nu-k+1}}}^{N-\nu\text{ columns}} & & & & \\
a'_0 & & & & b'_0 & & & & c'_0 & & & & \\
a'_1 & a'_0 & & & b'_1 & b'_0 & & & c'_1 & c'_0 & & & \\
\cdot & a'_1 & \cdot & & \cdot & b'_1 & \cdot & & \cdot & c'_1 & \cdot & & \\
\cdot & \cdot & \cdot\cdot & & \cdot & \cdot & \cdot\cdot & & \cdot & \cdot & \cdot\cdot & & N-k\text{ rows} \\
a'_{\lambda-k} & \cdot & \cdot\cdot & a'_0 & b'_{\mu-k} & \cdot & \cdot\cdot & b'_0 & \cdot & \cdot & \cdot\cdot & c'_0 & \\
 & a'_{\lambda-k} & \cdot\cdot & a'_1 & & b'_{\mu-k} & \cdot\cdot & b'_1 & \cdot & \cdot & \cdot\cdot & c'_1 & \\
 & & \cdots & & & & \cdots & & \cdot & \cdot & & \cdots & \\
 & & \cdot\cdot & & & & \cdot\cdot & & \cdot & \cdot & & \cdots & \\
 & & & a'_{\lambda-k} & & & & b'_{\mu-k} & \cdot & \cdot & & \cdots & \\
 & & & & & & & & c'_{N-k} & c'_{N-k-1} & \cdot\cdot & c'_{\nu-k+1} & \\
 & & & & & & & & \cdot & \cdot & & \cdots & k\text{ rows} \\
 & & & & & & & & \cdot & \cdot & & \cdots & \\
 & & & & & & & & c'_{N-1} & c'_{N-2} & \cdot\cdot & c'_\nu &
\end{array}$$

It is evident that each of the determinants of order N of the matrix $M'_N(F_1 F_2 F_3)$ which are not obviously equal to zero, is linearly expressible in terms of determinants of order k formed from the matrix

$$\begin{array}{||cccccc||l}
\overbrace{\hphantom{c'_{N-k+1}\;c'_{N-k}\;\cdot\;\cdot\;\cdot\;c'_{\nu-k+1}}}^{N-\nu\text{ columns}} & & & & & & \\
c'_{N-k} & c'_{N-k-1} & \cdot & \cdot & \cdot & c'_{\nu-k+1} & \\
c'_{N-k+1} & c'_{N-k} & \cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & k\text{ rows} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \\
c'_{N-1} & \cdot & & \cdot & \cdot & c'_\nu &
\end{array} \tag{18}$$

consisting of the last k rows of $M'_N(F_3)$.

But all of these determinants vanish by Lemma II of Chapter I, if the single determinant

$$D'_k = \begin{vmatrix}
c'_\nu & c'_{\nu-1} & \cdot & \cdot & c'_{\nu-k+1} \\
c'_{\nu+1} & c'_\nu & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
c'_{\nu+k-1} & \cdot & \cdot & \cdot & c'_\nu
\end{vmatrix}$$

formed from the last k columns of (18) vanishes. Therefore the matrix $M'_N(F_1 F_2 F_3)$ vanishes if $D'_k = 0$.

Now, the determinant of order N of $M'_N(F_1F_2F_3)$ formed from the $N-k$ columns containing $P_N^{(k)'}(F_r\,F_s)$ and the k columns containing D'_k is equal to $D'_k \,.\, P_N^{(k)'}(F_rF_s)$, and therefore its vanishing necessitates the vanishing of D'_k and therefore the vanishing of the matrix $M'_N(F_1F_2F_3)$. But this determinant is exactly the transform of $P_N^{(k)}(F_rF_s/F_j)$ by T_a. Hence, since this transformation does not change the rank of a matrix, the vanishing of $P_N^{(k)}(F_rF_s/F_j)$ necessitates the vanishing of $P_N^{(k)'}(F_rF_s/F'_j)$, therefore of the matrix $M'_N(F_1F_2F_3)$ and finally of the matrix $M_N(F_1F_2F_3)$. But by definition, the vanishing of $R^{(k)}(F_rF_s/F_j)$ is equivalent to the vanishing of $P_N^{(k)}(F_rF_s/F_j)$. Therefore the condition $R^{(k)}(F_rF_s/F_j)=0$, together with the assumed condition $R(F_rF_s)=0$, is equivalent to the vanishing of the matrix $M_N(F_1F_2F_3)$.

§ 9. *The Case in which a Sub-resultant Determinant $R^{(1)}(F_rF_s)$ is Different from Zero.*

THEOREM VI. *If F_r and F_s are any two of the polynomials* (4), *and if $R^{(1)}(F_rF_s) \neq 0$, then necessary and sufficient conditions for the vanishing of the matrix $M_N(F_1 \ldots . F_n)$, and therefore for the existence of a common factor of first or higher degree of the polynomials* (4), *are*

$$R(F_rF_s)=0,\quad R^{(1)}(F_rF_s/F_j)=0 \quad (j=1,2,\ldots,n, \text{ excepting } r,s). \tag{19}$$

The conditions are necessary, since the determinants in (19) are, when multiplied by certain powers of c_{r0}, equal to determinants of order N in the matrix $M_N(F_1 \ldots . F_n)$.

If now the determinant $R(F_rF_s)$ vanishes, then the polynomials F_r and F_s have a linear factor in common, and by means of a transformation T_a, the matrix $M_N(F_1 \ldots . F_n)$ can, according to Lemma III of Chapter II, be transformed into an equivalent matrix $M'_N(F_1 \ldots . F_n)$ such that the last row of the matrix $M'_N(F_rF_s)$ will consist wholly of zero elements. The groups of columns G_j $(j=1,2,\ldots,n$, excepting $r,s)$ will be transformed into

$$\left.\begin{array}{ccccc}
c'_{j0} & & & & \\
c'_{j1} & c'_{j0} & & & \\
\cdot & c'_{j1} & \cdot & & \\
\cdot & \cdot & \cdot & \cdot & \\
\cdot & \cdot & \cdot & \cdot & c'_{j0} \\
\cdot & \cdot & \cdot & \cdot & c'_{j1} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
c'_{jN-1} & c'_{jN-2} & \cdot & \cdot & c'_{jm_j}
\end{array}\right\} \begin{pmatrix} j=1,2,\ldots,n, \\ \text{excepting } r,s \end{pmatrix}.$$

Since now by hypothesis $R^{(1)}(F_r F_s)$ does not vanish, the same is true of the determinant $P_N^{(1)}(F_r F_s)$, and therefore of its transform $P_N^{(1)\prime}(F_r F_s)$. This latter determinant is, by definition, of order $N-1$; and by a well-known theorem in the theory of matrices, the matrix $M'_N(F_1 \ldots F_n)$ vanishes if every determinant of which $P_N^{(1)\prime}(F_r F_s)$ is a first minor vanishes. Certain of these determinants are obviously zero, namely those which are contained altogether in the matrix $M'_N(F_r F_s)$. The others are except for sign equal respectively to

$$c'_{jk} \cdot P_N^{(1)\prime}(F_r F_s) \quad \left(\begin{matrix} j = 1, 2, \ldots, n, \text{ excepting } r, s, \\ k = m_j,\ m_{j+1}, \ldots, N-1 \end{matrix}\right)$$

and vanish if and only if the quantities c'_{jk} vanish.

But by Lemma II of § 2, all the quantities c'_{jk} vanish if the quantities c'_{jm_j} vanish. Therefore the matrix $M'_N(F_1 \ldots F_n)$ vanishes if the $n-2$ determinants of that matrix which are equal to

$$c'_{jm_j} P_N^{(1)\prime}(F_r F_s) \quad (j = 1, 2, \ldots, n, \text{ excepting } r, s)$$

all vanish.

The latter determinants are, however, exactly the transforms of the determinants $P_N^{(1)}(F_r F_s/F_j)$ $(j = 1, 2, \ldots, n$, excepting $r, s)$; and therefore the vanishing of the determinants $P_N^{(1)}(F_r F_s/F_j)$, together with the vanishing of $R(F_r F_s)$, constitutes, under the hypothesis of the theorem, sufficient conditions for the vanishing of $M_N(F_1 \ldots F_n)$. Then since the vanishing of $R^{(1)}(F_r F_s/F_j)$ is from the definition of this determinant equivalent to the vanishing of $P^{(1)}(F_r F_s/F_j)$, it follows at once that conditions (19) are sufficient for the vanishing of $M_N(F_1 \ldots F_n)$.

Remark. It should be noted that if $R^{(1)}(F_r F_s)$ vanishes, then conditions (19) are by no means sufficient conditions for the vanishing of $M_N(F_1 \ldots F_n)$. In fact, if $R(F_r F_s) = R^{(1)}(F_r F_s) = 0$, then all of the determinants $R^{(1)}(F_r F_s/F_j)$ vanish whatever be the coefficients of F_j. For under these conditions the polynomials F_r and F_s have a quadratic factor in common, and therefore $M_N(F_1 \ldots F_n)$ can be transformed into an equivalent matrix $M''_N(F_1 \ldots F_n)$ such that the matrix $M''_N(F_r F_s)$ has only zero elements in the last *two* rows. Therefore the determinant $P_N^{(1)\prime\prime}(F_r F_s)$ vanishes, and so

$$c''_{jm_j} \cdot P_N^{(1)\prime\prime}(F_r F_s) = 0 \quad (j = 1, 2, \ldots, n, \text{ excepting } r, s). \tag{20}$$

But as before, the transforms of the determinants $P_N^{(1)}(F_r F_s/F_j)$ are exactly equal to the quantities on the left side of (20), and so $P_N^{(1)}(F_r F_s/F_j) = 0$, and therefore $R^{(1)} \cdot (F_r F_s/F_j) = 0$ $(j = 1, 2, \ldots, n$, excepting $r, s)$.

§ 10. *The General Case.*

THEOREM VII. *In any numerical case there are $n-1$ determinants of the matrix $M_N(F_1 \ldots . F_n)$ whose simultaneous vanishing is equivalent to the vanishing of the matrix, and therefore constitutes necessary and sufficient conditions for the existence of a common factor of first or higher degree of the polynomials* (4).

This theorem has been proved in § 8 for the case $n=3$; it will now be proved in general by a complete induction. We assume it true for $n=q$, and prove it true for $n=q+1$.

By our assumption, there are $q-1$ determinants of the matrix $M_N(F_1 \ldots F_q)$ whose vanishing is equivalent to the vanishing of the matrix. Let these determinants be denoted by

$$R(F_1F_2),\ R^{(k_3)}(F_1F_2/F_3),\ R^{(k_4)}(F_1F_2F_3/F_4),\ \ldots .,\ R^{(k_q)}(F_1F_2 \ldots . F_{q-1}/F_q),$$

and suppose that they all vanish. Then the matrix $M_N(F_1 \ldots . F_q)$ has a certain rank $N-k_{q+1}$, where $k_{q+1}>0$; the polynomials $F_1, F_2, \ldots ., F_q$ have a highest common factor of degree k_{q+1}; and there exists a determinant of order $N-2k_{q+1}$ of the sub-resultant matrix $M_N^{(k_{q+1})}(F_1 \ldots . F_q)$ which does not vanish.* Call it $R_N^{(k_{q+1})}(F_1 \ldots . F_q)$. There exists, therefore, a determinant of order $N-k_{q+1}$ formed from the first $N-k_{q+1}$ rows of $M_N(F_1 \ldots . F_q)$ which is equal to $c_{10}^{k_{q+1}} \cdot R^{(k_{q+1})}(F_1 \ldots . F_q)$ and so does not vanish. Call it $P_N^{(k_{q+1})}(F_1 \ldots . F_q)$.

Now, since the polynomials $F_1, F_2, \ldots ., F_q$ have a common factor of degree k_{q+1}, the matrix $M_N(F_1 \ldots . F_{q+1})$ can be changed by a transformation T_a into an equivalent matrix $M'_N(F_1 \ldots . F_{q+1})$ such that the last k_{q+1} rows of $M'_N(F_1 \ldots . F_q)$ contain only zero elements. By this same transformation the non-vanishing determinant $P_N^{(k_{q+1})}(F_1 \ldots . F_q)$ is changed into an equivalent determinant $P_N^{(k_{q+1})'}(F_1 \ldots . F_q)$.

For simplicity in writing, let

$$F_{q+1} \equiv c_0x^m + c_1x^{m-1} + \ldots . + c_m,$$

and for the moment let the number k_{q+1} be denoted simply by k. Then it is evident that each of the determinants of order N of the matrix $M'_N(F_1 \ldots . F_{q+1})$

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* There may of course be several such determinants. That there is at least one, follows from Theorems II and III.

which are not obviously zero, has as a factor a determinant of order k of the matrix

$$\begin{array}{cc} N-m \text{ columns} & \\ \left\|\begin{array}{cccccc} c'_{N-k} & c'_{N-k-1} & \cdot & \cdot & \cdot & c'_{m-k+1} \\ c'_{N-k+1} & c'_{N-k} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c'_{N-1} & c'_{N-2} & \cdot & \cdot & \cdot & c'_{m} \end{array}\right\| & k \text{ rows} \end{array} \qquad (21)$$

formed from the last k rows of the matrix $M'_N(F_{q+1})$. But by Lemma II the matrix (21) vanishes if the single determinant

$$D'_k \equiv \left|\begin{array}{ccccc} c'_m & c'_{m-1} & \cdot & \cdot & c'_{m-k+1} \\ c'_{m+1} & c'_m & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c'_{m+k-1} & \cdot & \cdot & \cdot & c'_m \end{array}\right|$$

formed from the last k columns of (21) vanishes. Therefore the matrix $M'_N(F_1 \ldots . F_{q+1})$ vanishes if $D'_k = 0$.

Now the determinant of order N of $M'_N(F_1 \ldots . F_{q+1})$ formed from the $N-k$ columns containing $P^{(k)\prime}_N(F_1 \ldots . F_q)$ and the k columns containing D'_k is equal to $D'_k \cdot P^{(k)\prime}_N(F_1 \ldots . F_q)$. Therefore its vanishing necessitates the vanishing of D'_k, and so of $M'_N(F_1 \ldots . F_{q+1})$. This determinant, which may be denoted by $P^{(k)\prime}_N(F_1 \ldots . F_q/F_{q+1})$, is the transform by T_α of a determinant of $M_N(F_1 \ldots . F_{q+1})$, which may be denoted by $P^{(k)}_N(F_1 \ldots . F_q/F_{q+1})$. Therefore, since the transformation T_α does not change the rank of a matrix, the vanishing of $P^{(k)}_N(F_1 \ldots . F_q/F_{q+1})$ necessitates the vanishing of $P^{(k)\prime}_N(F_1 \ldots . F_q/F_{q+1})$, therefore of the matrix $M'_N(F_1 \ldots . F_{q+1})$, and finally of $M_N(F_1 \ldots . F_{q+1})$.

But the determinant $P^{(k)}_N(F_1 \ldots . F_q/F_{q+1})$ contains as a factor a certain power of c_{10}, say c^p_{10}. The determinant of order $N-p$ which is equal to the quotient $1/c^p_{10} \cdot P^{(k)}_N(F_1 \ldots . F_q/F_{q+1})$ is denoted by $R^{(k)}(F_1 \ldots . F_q/F_{q+1})$, and evidently vanishes only if $P^{(k)}_N(F_1 \ldots . F_q/F_{q+1})$ vanishes.

Replacing the index k by its equal k_{q+1}, we have proved that the vanishing of the determinants

$$\left.\begin{array}{c} R(F_1 F_2), \quad R^{(k_3)}(F_1 F_2/F_3), \quad \ldots ., \quad R^{(k_q)}(F_1 \ldots . F_{q-1}/F_q), \\ R^{(k_{q+1})}(F_1 \ldots . F_q/F_{q+1}) \end{array}\right\} \qquad (22)$$

is sufficient for the vanishing of the matrix $M_N(F_1 \ldots . F_{q+1})$. As the vanishing of the $n-1$ determinants (22) is evidently necessary, the theorem is completely proved.

CHAPTER IV.

An Illustrative Example. Three Cubics.

Given

$$\left.\begin{aligned} F_1(x) &\equiv a_0x^3 + a_1x^2 + a_2x + a_3, \\ F_2(x) &\equiv b_0x^3 + b_1x^2 + b_2x + b_3, \\ F_3(x) &\equiv c_0x^3 + c_1x^2 + c_2x + c_3. \end{aligned}\right\} \tag{23}$$

The resultant matrix of lowest possible order is

$$M_6(F_1F_2F_3) \equiv \left\| \begin{array}{ccccccccc} a_0 & & & b_0 & & & c_0 & & \\ a_1 & a_0 & & b_1 & b_0 & & c_1 & c_0 & \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ & a_3 & a_2 & & b_3 & b_2 & & c_3 & c_2 \\ & & a_3 & & & b_3 & & & c_3 \end{array} \right\|$$

Its vanishing is necessary and sufficient for the existence of a common factor of first or higher degree of F_1, F_2 and F_3.

The two resultant determinants can now be constructed, their form depending, however, upon the values of the sub-resultants of some pair of the polynomials (see Theorem V). For definiteness, we consider the sub-resultants of the pair F_1 and F_2. We have by definition

$$R(F_1F_2) \equiv \left| \begin{array}{cccccc} a_0 & & & b_0 & & \\ a_1 & a_0 & & b_1 & b_0 & \\ a_2 & a_1 & a_0 & b_2 & b_1 & b_0 \\ a_3 & a_2 & a_1 & b_3 & b_2 & b_1 \\ & a_3 & a_2 & & b_3 & b_2 \\ & & a_3 & & & b_3 \end{array} \right|$$

$$R^{(1)}(F_1F_2) \equiv \left| \begin{array}{cccc} a_0 & & b_0 & \\ a_1 & a_0 & b_1 & b_0 \\ a_2 & a_1 & b_2 & b_1 \\ a_3 & a_2 & b_3 & b_2 \end{array} \right| \qquad R^{(2)}(F_1F_2) \equiv \left| \begin{array}{cc} a_0 & b_0 \\ a_1 & b_1 \end{array} \right|$$

There are then three distinct cases to be considered.

Case I: *In which* $R^{(1)}(F_1F_2) \neq 0$. The two resultant determinants whose vanishing is equivalent to the vanishing of $M_6(F_1F_2F_3)$ are

$$R(F_1F_2), \quad R^{(1)}(F_1F_2/F_3) \equiv \left| \begin{array}{ccccc} a_0 & & b_0 & & \\ a_1 & a_0 & b_1 & b_0 & c_0 \\ a_2 & a_1 & b_2 & b_1 & c_1 \\ a_3 & a_2 & b_3 & b_2 & c_2 \\ & a_3 & & b_3 & c_3 \end{array} \right|$$

Case II: *In which* $R^{(1)}(F_1 F_2) = 0$, $R^{(2)}(F_1 F_2) \neq 0$. The resultant determinants are

$$R(F_1 F_2), \quad R^{(2)}(F_1 F_2/F_3) \equiv \begin{vmatrix} a_0 & & & c_0 & \\ a_1 & a_0 & b_0 & c_1 & c_0 \\ a_2 & a_1 & b_1 & c_2 & c_1 \\ a_3 & a_2 & b_2 & c_3 & c_2 \\ & a_3 & b_3 & & c_3 \end{vmatrix}$$

Case III: *In which* $R^{(1)}(F_1 F_2) = R^{(2)}(F_1 F_2) = 0$, $R^{(3)}(F_1 F_2) \neq 0$. The resultant determinants are

$$R(F_1 F_2), \quad R^{(3)}(F_1 F_2/F_3) \equiv \begin{vmatrix} b_0 & & & c_0 & & \\ b_1 & b_0 & & c_1 & c_0 & \\ b_2 & b_1 & b_0 & c_2 & c_1 & c_0 \\ b_3 & b_2 & b_1 & c_3 & c_2 & c_1 \\ & b_3 & b_2 & & c_3 & c_2 \\ & & b_3 & & & c_3 \end{vmatrix}$$

In this case, if $R(F_1 F_2) = 0$, then the polynomials F_1 and F_2 differ only by a constant factor. Also $R^{(3)}(F_1 F_2/F_3)$ is identical with $R(F_1 F_3)$. It is therefore directly evident that the vanishing of $R(F_1 F_2)$ and $R^{(3)}(F_1 F_2/F_3)$ is sufficient for the existence of a common factor of first or higher degree of F_1, F_2 and F_3.

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